NORTH SYDNEY BOYS HIGH SCHOOL

MATHEMATICS (EXTENSION 2)

2012 HSC Course Assessment Task 4

August 17, 2012

General instructions

- Working time – 50 min.
- **Commence each new question on a new page**.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

Class (please ✔)

- ◯ 12M4A – Mr Lin
- ◯ 12M4B – Mr Ireland
- ◯ 12M4C – Mr Fletcher

STUDENT NUMBER ....

 # BOOKLETS USED: ....

Marker's use only.

QUESTION	1	2	3	**Total**	%
MARKS	$\overline{12}$	$\overline{11}$	$\overline{11}$	$\overline{34}$	

Glossary

- $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3\}$ – set of all integers.
- $\mathbb{Z}^+$ – all positive integers (excludes zero)
- $\mathbb{R}$ – set of all real numbers

Question 1 (12 Marks) Commence a NEW page. **Marks**

(a) Given that $a^2 + b^2 \geq 2ab$ and $a^2 + b^2 + c^2 \geq ab + bc + ac$, where a, b and $c \in \mathbb{R}^+$, show that:

i. $\sin^2\alpha + \cos^2\alpha \geq \sin 2\alpha$. **1**

ii. $\sin^2\alpha + \cos^2\alpha + \tan^2\alpha \geq \sin\alpha - \cos\alpha + \sec\alpha + \dfrac{1}{2}\sin 2\alpha$. **3**

(b) i. Show that $\dbinom{n}{r} < n\dbinom{n-1}{r-1}$, where n, $r \in \mathbb{Z}^+$ and $1 < r \leq n$. **2**

ii. Given that $(a+b)^n$ can be written as **2**

$$\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n}b^n$$

Show that for a, $b > 0$ and $n \in \mathbb{Z}^+$,

$$(a+b)^n - a^n < nb(a+b)^{n-1}$$

(c) Show that $x > \dfrac{3\sin x}{2 + \cos x}$ for $x > 0$. **4**

Question 2 (11 Marks) Commence a NEW page. **Marks**

A particle of mass m kg is dropped from rest in a medium in which the resistance to motion has magnitude $\frac{1}{10}mv^2$ when the velocity of the particle is v ms^{-1}.

After t seconds, the particle has fallen x metres and has velocity v ms^{-1} and acceleration a ms^{-2}. Take the acceleration due to gravity as $10\,\text{ms}^{-2}$.

(a) Draw a diagram showing the forces acting on the particle. Hence show that **1**

$$a = \frac{100 - v^2}{10}$$

(b) Show that $t = \dfrac{1}{2}\log_e\left(\dfrac{10+v}{10-v}\right)$. **3**

(c) Find expressions in terms of t for v and x. **4**

(d) Show that the terminal velocity is $10\,\text{ms}^{-1}$. **1**

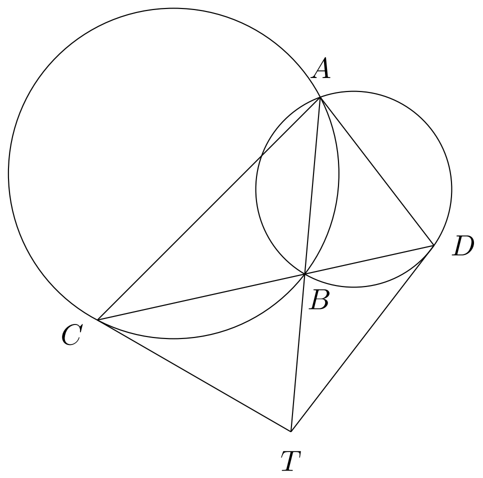
(e) Find the exact time taken and the exact distance fallen by the particle in reaching a speed equal to 80% of its terminal velocity. **2**

Question 3 (11 Marks) Commence a NEW page. **Marks**

(a) Use mathematical induction to prove for $n \geq 1$, $n \in \mathbb{Z}^+$: **3**

$$1 \times 1! + 2 \times 2! + \cdots + n \times n! = (n+1)! - 1$$

(b) BAC, BAD are two circles such that tangents at C and D meet at T on AB produced. If CBD is a straight line, prove that



i. $TC = TD$. **1**

ii. $\angle TAC = \angle TAD$. **2**

iii. $TCAD$ is a cyclic quadrilateral. **2**

(c) The number f_n is defined as $f_n = 2^{2^n} + 1$ for $n \in \mathbb{Z}^+$, where 2^{2^n} is 2 raised to the power of 2^n. **3**

Prove, using mathematical induction, that for all $n \in \mathbb{Z}^+$,

$$f_0 f_1 f_2 \cdots f_{n-1} = f_n - 2$$

End of paper.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \qquad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x + C, \qquad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C, \qquad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C, \qquad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C, \qquad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C, \qquad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + C, \qquad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \qquad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + C, \qquad a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \qquad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE: $\ln x = \log_e x, \; x > 0$

QUESTION 1

(a) (i) Let $a = \sin\alpha$ and $b = \cos\alpha$

then $\sin^2\alpha + \cos^2\alpha \geq 2\sin\alpha\cos\alpha$

i.e. $\sin^2\alpha + \cos^2\alpha \geq \sin 2\alpha$

1 mark for correct proof

(ii) Let $a = \sin\alpha$, $b = \cos\alpha$, $c = \tan\alpha$

then $\sin^2\alpha + \cos^2\alpha + \tan^2\alpha \geq \sin\alpha\cos\alpha + \sin\alpha\tan\alpha + \cos\alpha\tan\alpha$

1 mark

$$RHS = \frac{1}{2}\sin 2\alpha + \sin\alpha + \frac{\sin^2\alpha}{\cos\alpha}$$

~~1 mark~~

$$= \frac{1}{2}\sin 2\alpha + \sin\alpha + \frac{1-\cos^2\alpha}{\cos\alpha}$$

1 mark

$$= \frac{1}{2}\sin 2\alpha + \sin\alpha + \sec\alpha - \cos\alpha$$

1 mark

(b) (i)
$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)!}{r(r-1)!(n-r)!}$$

1 mark

$$= \frac{1}{r} \times n \times \binom{n-1}{r-1}$$

$$< n\binom{n-1}{r-1} \quad \text{because } r > 1 \quad \therefore \frac{1}{r} < 1$$

1 mark

(ii)
$$(a+b)^n - a^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n}b^n - a^n$$

$$= \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n$$

$$< n\binom{n-1}{0}a^{n-1}b + n\binom{n-1}{1}a^{n-2}b^2 + \cdots + n\binom{n-1}{n-1}b^n$$

from (i)

1 mark

$$= nb\left[\binom{n-1}{0}a^{n-1} + \binom{n-1}{1}a^{n-2}b + \cdots + \binom{n-1}{n-1}b^{n-1}\right]$$

$$= nb(a+b)^{n-1}$$

1 mark

i.e. $(a+b)^n - a^n < nb(a+b)^{n-1}$

(c) Let $f(x) = x - \dfrac{3\sin x}{2+\cos x}$

then $f'(x) = 1 - \dfrac{(2+\cos x)3\cos x - 3\sin x \cdot -\sin x}{(2+\cos x)^2}$

$= 1 - \dfrac{6\cos x + 3\cos^2 x + 3\sin^2 x}{(2+\cos x)^2}$

$= \dfrac{4 + 4\cos x + \cos^2 x - 6\cos x - 3}{(2+\cos x)^2}$

$= \dfrac{1 - 2\cos x + \cos^2 x}{(2+\cos x)^2}$

$= \left(\dfrac{1-\cos x}{2+\cos x}\right)^2$ 1 mark

$f'(x) \geq 0$ for $x > 0$

$f'(0) = 0$ and $f'(2n\pi) = 0$ ~~1 mark~~

$\therefore$ $f(x)$ is non-decreasing for $x > 0$ 1 mark

$f(0) = 0$ and $f(2n\pi) > 0$ 1 mark

$\therefore$ $f(x) > 0$ for $x > 0$ 1 mark

ie $x > \dfrac{3\sin x}{2+\cos x}$ for $x > 0$

QUESTION 2

(a) $\uparrow \frac{1}{10}mv^2$ $\downarrow 10m$

$$ma = 10m - \frac{1}{10}mv^2$$
$$\therefore a = 10 - \frac{1}{10}v^2$$
$$= \frac{100-v^2}{10}$$

1 mark

(b)
$$\frac{dv}{dt} = \frac{100-v^2}{10}$$
$$\frac{dt}{dv} = \frac{10}{100-v^2} = \frac{10}{(10+v)(10-v)}$$
$$= \frac{1}{2}\left(\frac{1}{10+v} + \frac{1}{10-v}\right)$$
using partial fractions

$$\frac{10}{(10+v)(10-v)} = \frac{b}{10+v} + \frac{c}{10-v}$$
$$10 = b(10-v) + c(10-v)$$
sub. $v = 10 \Rightarrow c = \frac{1}{2}$
sub $v = -10 \Rightarrow b = \frac{1}{2}$

1 mark for reducing to partial fractions

$$\therefore t = \frac{1}{2}\left(\ln(10+v) - \ln(10-v)\right) + k$$

1 mark for integration

when $t=0$, $v=0$ $\therefore k=0$
$$\therefore t = \frac{1}{2}\ln\left(\frac{10+v}{10-v}\right)$$

1 mark for evaluating constant

(c)
$$e^{2t} = \frac{10+v}{10-v}$$

1 mark for e^{2t}

$$10e^{2t} - ve^{2t} = 10+v$$
$$v = \frac{10(e^{2t}-1)}{e^{2t}+1}$$

1 mark for v

$$\frac{dx}{dt} = \frac{10(e^{t}-e^{-t})}{e^{t}+e^{-t}}$$

1 mark for $\frac{dx}{dt}$

$$x = 10\ln(e^{t}+e^{-t}) + c$$
when $t=0$, $x=0$ $\therefore c = -10\ln 2$
$$\therefore x = 10\ln\left(\frac{e^{t}+e^{-t}}{2}\right)$$

1 mark for x

(d) as $a \to 0$, $\frac{100-v^2}{10} \to 0$
$$v^2 = 100$$
$$v = 10 \quad (v>0)$$

1 mark

(e)
$$v = 8 \Rightarrow t = \frac{1}{2}\ln\left(\frac{18}{2}\right)$$
$$= \frac{1}{2}\ln 9$$
$$= \ln 3$$

1 mark for any of these expressions

$$x = 10\ln\left(\frac{e^{\ln 3}+e^{-\ln 3}}{2}\right)$$
$$= 10\ln\left(\frac{3+\frac{1}{3}}{2}\right)$$
$$= 10\ln\left(\frac{5}{3}\right)$$

1 mark for any of these expressions

QUESTION 3

(a) When $n=1$, LHS $= 1 \times 1! = 1$

RHS $= 2! - 1 = 2 - 1 = 1$

LHS = RHS

$\therefore$ true for $n=1$ — 1 mark for proving true for $n=1$

Assume true for $n=k$

i.e. $1\times1! + 2\times2! + \dots + k \times k! = (k+1)! - 1$

Let $n=k+1$, LHS $= 1\times1! + 2\times2! + \dots + k\times k! + (k+1)\times(k+1)!$

$= (k+1)! - 1 + (k+1)(k+1)!$ from assumption — 1 mark for correctly substituting from assumption

$= (k+1)!\,(1 + k+1) - 1$

$= (k+2)(k+1)! - 1$

$= (k+2)! - 1 =$ RHS — 1 mark for correct proof for $n=k+1$

i.e. true for $n=k+1$ if true for $n=k$

But proved true for $n=1$

$\therefore$ true for $n=2,3,4,\dots$ and all positive integers

(b) (i) $TC^2 = TB \cdot TA$ (Square of tangent equals product of intercepts of ~~to~~ secant from the same point)

$TD^2 = TB \cdot TA$

i.e. $TC^2 = TD^2$

$\therefore TC = TD$ — 1 mark – reason not necessary

(ii) Then $\angle TCD = \angle TDC$ (isos. ΔTCD) — 2 marks for correct proof

$\angle TCD = \angle TAC$ (angle between tangent and chord equals angle in alternate segment) — −1 mark for ~~[illegible]~~ no reasons or incorrect reasons.

and $\angle TDC = \angle TAD$

$\therefore \angle TAC = \angle TAD$

(iii) ~~From~~ (ii) $\angle TCD = \angle TAC = \angle TAD$ — 1 mark

$\therefore$ TCAD is a cyclic quadrilateral as the angles in the same segment at C and A subtended from TD are equal — 1 mark

(c) When $n=1$, $LHS = f_0 = 2^{2^0}+1 = 2^1+1 = 3$

$RHS = f_1 - 2 = 2^{2^1}+1-2 = 2^2+1-2 = 3$

$LHS = RHS$

$\therefore$ true for $n=1$

1 mark for proving true for $n=1$

Assume true for $n=k$

i.e. $f_0 f_1 \cdots f_{k-1} = f_k - 2$

Let $n=k+1$,

$$\begin{aligned}
LHS &= f_0 f_1 \cdots f_{k-1} f_k \\
&= (f_k - 2) f_k \quad \text{from assumption} \\
&= (2^{2^k}+1-2)(2^{2^k}+1) \\
&= (2^{2^k}-1)(2^{2^k}+1) \\
&= (2^{2^k})^2 - 1 \\
&= 2^{2^k \times 2} - 1 \\
&= 2^{2^{k+1}} - 1 \\
&= 2^{2^{k+1}} + 1 - 2 = f_{k+1} - 2 \\
&= RHS
\end{aligned}$$

2 marks for correct proof

1 mark for correct substitution from assumption

i.e. true for $n=k+1$ if true for $n=k$

But true for $n=1$, $\therefore$ true for $n=2,3,4,\ldots$ and all positive integers